11 The field C of complex numbers

Before continuing with linear algebra, I must spend some time talking about another important field, namely field of the complex numbers $\mathbf{C}$. The motivation for this is the necessity to find roots of a given polynomial. We all well aware that not every polynomial has real roots (think about $f(x) = x^2 + 1$ for instance). On the other hand, *the fundamental theorem of algebra* says that any polynomial of degree at least one has at least one complex root, which will be most important for us.

11.1 Naive introduction to complex numbers

The usual way to quickly start with complex numbers is to say that we introduce a new object, which I will call i (engineers very often use j), such that the following *characteristic* property holds:

$$\mathrm{i}^2 = -1.$$

Now, having this new object at my disposal, I can easily define the set of complex numbers as the set with elements of the form $x + \mathrm{i}y$, where x, y are our familiar real numbers:

$$\mathbf{C} := \{x + \mathrm{i}y \colon x, y \in \mathbf{R},\ \mathrm{i}^2 = -1\}.$$

After this, using the usual rules of the arithmetic operations as applied to the real numbers, I have

$$\begin{aligned} z_1 \pm z_2 &= (x_1 + \mathrm{i}x_2) \pm (y_1 + \mathrm{i}y_2) = (x_1 \pm y_1) + \mathrm{i}(x_2 \pm y_2), \\ z_1 z_2 &= (x_1 + \mathrm{i}x_2)(y_1 + \mathrm{i}y_2) = (x_1y_1 - x_2y_2) + \mathrm{i}(x_1y_2 + x_2y_1), \\ \frac{z_1}{z_2} &= \frac{x_1 + \mathrm{i}x_2}{y_1 + \mathrm{i}y_2} = \frac{(x_1 + \mathrm{i}x_2)(y_1 - \mathrm{i}y_2)}{(y_1 + \mathrm{i}y_2)(y_1 - \mathrm{i}y_2)} = \frac{x_1y_1 + x_2y_2}{y_1^2 + y_2^2} + \mathrm{i}\frac{x_2y_1 - x_1y_2}{y_1^2 + y_2^2}, \end{aligned}$$

and therefore I have arithmetics in the set of complex numbers.

The used here definition is, however, not really satisfactory, because it *does not explain* what i is, and hence we get a sense of mystery here (and hence the name "imaginary" numbers). But there is actually nothing imaginary about complex numbers, as was realized by Gauss and others, if we identify them with the elements of our familiar $\mathbf{R}^2$ and add a little more. Therefore, let me introduce the set of complex numbers in a different and rigorous way (as a side remark I note that there are other ways to define complex numbers, but the one I am using is arguably the most natural).

11.2 Definition and basic properties

Definition 11.1. *The set of complex numbers, which is denotes* $\mathbf{C}$*, is, by definition,* $\mathbf{R}^2$*, that is, the set of all the vectors with two real coordinates, on which the operations of addition and multiplication are defined as follows (to save my space I write my vectors as row-vectors):*

$$\begin{aligned} (x_1, x_2) + (y_1, y_2) &= (x_1 + y_1, x_2 + y_2), \\ (x_1, x_2) \cdot (y_1, y_2) &= (x_1y_1 - x_2y_2, x_1y_2 + x_2y_1). \end{aligned}$$

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e-mail: artem.novozhilov@ndsu.edu. Fall 2016

Definitely, the multiplication rule should look a little strange at the beginning, but of course the motivation comes from the previous subsection. Before getting any intuition, let me prove

Proposition 11.2. *The set* $\mathbf{C}$ *with defined in this way addition and multiplication is a field. That is, for any* $z_1, z_2, z_3 \in \mathbf{C}$ *we have*

$$\begin{aligned} z_1 + z_2 &= z_2 + z_1, \\ (z_1 + z_2) + z_3 &= z_1 + (z_2 + z_3), \\ z_1 z_2 &= z_2 z_1, \\ (z_1 z_2) z_3 &= z_1 (z_2 z_3), \end{aligned}$$

there is a unique element $0 \in \mathbf{C}$ *for which*

$$z_1 + 0 = z_1,$$

for any $z_1 \in \mathbf{C}$ *there is* $-z_1 \in \mathbf{C}$ *for which*

$$z_1 + (-z_1) = 0,$$

there is $1 \in \mathbf{C}$*, for which*

$$1 z_1 = z_1,$$

and finally for any nonzero $z \in \mathbf{C}$ *there is* $z^{-1} \in \mathbf{C}$ *for which*

$$z z^{-1} = 1.$$

Remark 11.3. It should be clear that above $0 = (0, 0)$, where 0 on the left is the zero of $\mathbf{C}$ and zero on the right is our familiar real zero; also, $1 \in \mathbf{C}$ means that $1 = (1, 0)$. Probably the only non-trivial fact to prove is to show the existence of the multiplicative inverse. I again use the informal calculations from the introductory subsection, and hence here is an explicit formula for it: If $z = (x, y) \in \mathbf{C}$ then

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right).$$

Exercise 1. Prove all the points of the proposition above.

Now, consider the following equation

$$z^2 + 1 = 0.$$

Here 1 and 0 are complex unit and zero. We all know that this equation has no real roots. But I claim that it has two complex roots $(0, 1)$ and $-(0, 1)$ (check this!). To make my life much easier I will *define* now one of the roots of my equation to be *the imaginary unit* $\mathrm{i} = (1, 0)$, and my equation actually shows that $\mathrm{i}^2 = -1$. Now any complex number can be represented as $z = x + \mathrm{i}y = 1x + \mathrm{i}y$, where $1 = (1, 0)$ (of course you should recognize that complex 1 and i form the standard basis of $\mathbf{R}^2$), and I will tacitly assume the $\mathbf{R} \subseteq \mathbf{C}$ (formally, of course, it is not true, but I can always show that $\mathbf{R}$ is isomorphic to a subset of $\mathbf{C}$, think this out).

Now, having at my disposal a rigorous definition of the field of complex numbers, I actually *justified* the calculations like the following one:

$$z_1 \cdot z_2 = (x_1 + \mathrm{i}x_2)(y_1 + \mathrm{i}y_2) = x_1y_1 + \mathrm{i}^2 x_2y_2 + \mathrm{i}(x_2y_1 + x_1y_2) = x_1y_1 - x_2y_2 + \mathrm{i}(x_2y_1 + x_1y_2).$$

A little more definitions: For any complex $z = x + \mathrm{i}y \in \mathbf{C}$ x is called the real part of z and denoted $\operatorname{Re} z$, and y is called the imaginary part of z and denoted $\operatorname{Im} z$. Thus $z = \operatorname{Re} z + \mathrm{i} \operatorname{Im} z$. The *modulus* or *an absolute value* of z is $(x^2 + y^2)^{1/2}$ and denoted $|z|$. Finally, complex *conjugate* of z is $\bar{z} = x - \mathrm{i}y$.

Exercise 2. Prove that

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2.$$

Note that according to the introduced definitions $|z|^2 = z\bar{z}$.

Exercise 3. Show that $|z_1 z_2| = |z_1||z_2|$.

Using the complex conjugate we can easily find a multiplicative inverse:

$$z^{-1} = \frac{1}{z} = \frac{1}{x + \mathrm{i}y} = \frac{x - \mathrm{i}y}{(x + \mathrm{i}y)(x - \mathrm{i}y)} = \frac{\operatorname{Re} z}{|z|^2} + \mathrm{i}\frac{-\operatorname{Im} z}{|z|^2},$$

which gives a clue how the formula for the inverse was actually found.

Using the geometry of $\mathbf{R}^2$ I can represent complex numbers using the polar coordinates, see the figure below.

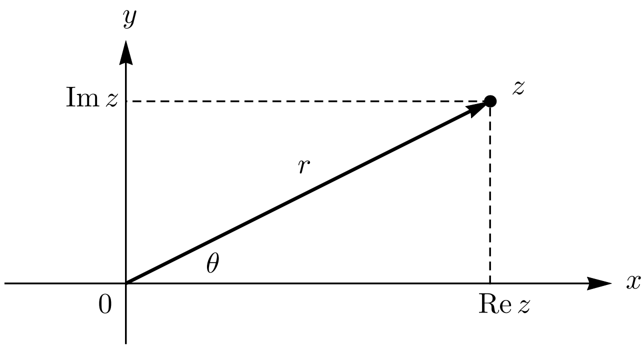

Figure 1: Geometric interpretation of complex numbers. θ is the angle between vector z and x-axis; r is the length of the same vector: $r = |z|$

Since $(x, y) = (r\cos\theta, r\sin\theta)$, where $r = |z|$ and $\tan\theta = y/x$ (for $x \neq 0$), then

$$z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta),$$

which is called *trigonometric form* of complex number. The angle θ is called an *argument* of z and denoted $\arg z$, note that the argument is defined only modulo 2π and is not unique. It is convenient also to have the principal value of the polar angle, which is denoted $\operatorname{Arg} z$ and satisfies $0 \leq \operatorname{Arg} z < 2\pi$ (or, sometimes, $-\pi < \operatorname{Arg} z \leq \pi$).

Convince yourself that

$$|\mathrm{i}| = 1,\ \mathrm{Arg}\,\mathrm{i} = \frac{\pi}{2}\,,\quad |1+\mathrm{i}| = \sqrt{2},\ \mathrm{Arg}(1+\mathrm{i}) = \frac{\pi}{2}\,.$$

Using trigonometric form it is possible (do it!) to show that if we are given two complex numbers z_1 and z_2 then their product has the modulus equal to $|z_1||z_2|$ and argument $\theta_1 + \theta_2$.

Exercise 4. Since the principle argument of i is $\pi/2$ and modulus is 1, then, from the above, the multiplication by i amounts to rotating the given vector by the angle $\pi/2$. Since we know that it is a linear operator acting on $\mathbf{R}^2$ it has a matrix. Find it.

11.3 Euler's formula

For any complex z it is true that

$$e^{\mathrm{i}z} = \cos z + \mathrm{i}\sin z.$$

This equality is called *Euler's formula*, but to fully appreciate it I would need to discuss what the function of the complex argument is, and this is beyond the scope of this course. Instead, I will talk about a particular case, which is true for any $x \in \mathbf{R}$:

$$e^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x.$$

To get an idea where this remarkable identity coming from, recall that functions e^x, $\cos x$, $\sin x$ have Taylor's series absolutely convergent for any $x \in \mathbf{R}$:

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}\,,\\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}\,,\\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!}\,.
\end{aligned}$$

Now plug $\mathrm{i}x$ instead of x into the series for the exponent function, use the property that $\mathrm{i}^2 = -1$, $\mathrm{i}^3 = -\mathrm{i}$, $\mathrm{i}^4 = 1, \ldots$, rearrange the series and obtain Euler's formula.

Using Euler's formula we can write any complex number in the *exponential form* as

$$z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta) = re^{\mathrm{i}\theta}.$$

Using Euler's formula we also can express usual cos and sin functions through the exponent:

$$\cos x = \frac{e^{\mathrm{i}x} + e^{-\mathrm{i}x}}{2}\,,\quad \sin x = \frac{e^{\mathrm{i}x} - e^{-\mathrm{i}x}}{2\mathrm{i}}\,.$$

The exponential form of the complex number lets us obtain a number of results. The one is that

$$z^n = (re^{\mathrm{i}\theta})^n = r^n e^{\mathrm{i}n\theta} = r^n(\cos n\theta + \mathrm{i}\sin n\theta),$$

which is called *De Moivre's formula* if $r = 1$:

$$(\cos\theta + \mathrm{i}\sin\theta)^n = (\cos n\theta + \mathrm{i}\sin n\theta).$$

Example 11.4. What is $(1+\mathrm{i})^6$? I can, of course, multiply the factor $(1+\mathrm{i})$ six times (or use the binomial theorem). However, a better approach would be to consider

$$1+\mathrm{i}=\sqrt{2}e^{\mathrm{i}\frac{\pi}{4}} \implies (1+\mathrm{i})^6=(\sqrt{2})^6e^{\mathrm{i}\frac{3\pi}{2}}=8\mathrm{i}.$$

For an arbitrary $z=x+\mathrm{i}y$ I have

$$e^{x+\mathrm{i}y}=e^xe^{\mathrm{i}y}=e^x(\cos y+\mathrm{i}\sin y).$$

Hence

$$\operatorname{Re} e^z=e^x\cos y,\quad \operatorname{Im} e^z=e^x\sin y.$$

Example 11.5. Find the formula for $\cos^3 x$. I can use

$$\cos x=\frac{e^{\mathrm{i}x}+e^{-\mathrm{i}x}}{2},$$

therefore

$$\cos^3 x=\frac{1}{8}(e^{\mathrm{i}x}+e^{-\mathrm{i}x})^3=\frac{1}{8}(e^{3\mathrm{i}x}+3e^{\mathrm{i}x}+3e^{-\mathrm{i}x}+e^{-3\mathrm{i}x}),$$

which can be rewritten as

$$\cos^3 x=\frac{1}{4}\cos 3x+\frac{3}{4}\cos x.$$

You should do the same calculations for $\sin^3 x$.

11.4 The fundamental theorem of algebra

The complex numbers appeared while solving polynomial equations. Consider a polynomial P_n of complex variable z with complex coefficients:

$$P_n(z)=z^n+c_1z^{n-1}+c_2z^{n-2}+\ldots+c_{n-1}z+c_n,$$

where $c_j\in\mathbf{C}$. Any number $\hat{z}\in\mathbf{C}$ such that $P_n(\hat{z})=0$ is called a *root* of $P_n(z)$. The most important fact here is called the *fundamental theorem of algebra*:

Theorem 11.6. *Any complex polynomial of degree at least one has at least one complex root.*

If you are interested in seing a proof of this remarkable theorem, you should consider taking MATH 452.

Using the fundamental theorem of algebra it is quite straightforward to prove that any complex polynomial of degree n can be factored into the following form:

$$P_n(z)=(z-\hat{z}_1)^{a_1}(z-\hat{z}_2)^{a_2}\ldots(z-\hat{z}_k)^{a_k},\quad a_1+a_2+\ldots+a_k=n,\ a_j\in\mathbf{N},$$

where $\hat{z}_j$ are the roots of P_n. The integers a_j are called (algebraic) multiplicities of the roots, and therefore this result can be rephrased as "any complex polynomial of degree n has exactly n complex roots counting their multiplicities." In a more algebraic language, one say that "the field of complex numbers is algebraically closed."

Example 11.7. Factor z^2+1. Since the roots of this polynomial are $\hat{z}_{1,2}=\pm\mathrm{i}$ then

$$z^2+1=(z-\mathrm{i})(z+\mathrm{i}).$$

Example 11.8. Solve $z^{10}+1=0$. This basically means that I am asked to find 10 roots of the 10-th degree from -1. To do this I will use the exponential form of the complex numbers. I obviously have $-1=1e^{\mathrm{i}\frac{3\pi}{2}}$. If my unknown root has the form $\hat{z}=re^{\mathrm{i}\theta}$ then my equation reads

$$r^{10}e^{\mathrm{i}(10\theta)}=1e^{\mathrm{i}\frac{3\pi}{2}},$$

or

$$r=1,\quad 10\theta=\frac{3\pi}{2}+2\pi k,\quad k\in\mathbf{Z}.$$

Hence I can conclude that my ten roots have the arguments

$$\theta_k=\frac{3\pi}{20}+\frac{2\pi k}{10},\quad k=0,\dots,9,$$

since for $k=10$ I will get exactly the same θ as for $k=0$.

Note that geometrically the solutions to this problem are the vertices of a right polygon with 10 vertices, which is sometimes very convenient for quick geometric calculations.

Exercise 5. Consider a polynomial with real coefficients:

$$P_n(x)=x^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0,\quad a_j\in\mathbf{R}.$$

Show that if $z\in\mathbf{C}$ is a root of P_n then $\bar{z}\in\mathbf{C}$ is also its root.